

A NOTE ON THE COMPUTABLE CATEGORICITY OF ℓ^p SPACES

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ABSTRACT. Suppose that p is a computable real and that $p \geq 1$. We show that in both the real and complex case, ℓ^p is computably categorical if and only if $p = 2$. The proof uses Lamperti's characterization of the isometries of Lebesgue spaces of σ -finite measure spaces.

1. INTRODUCTION

When p is a positive real number, let ℓ^p denote the space of all sequences of complex numbers $\{a_n\}_{n=0}^\infty$ so that

$$\sum_{n=0}^{\infty} |a_n|^p < \infty.$$

ℓ^p is a vector space over \mathbb{C} with the usual scalar multiplication and vector addition. When $p \geq 1$ it is a Banach space under the norm defined by

$$\|\{a_n\}_n\| = \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{1/p}.$$

Loosely speaking, a computable structure is *computably categorical* if all of its computable copies are computably isomorphic. In 1989, Pour-El and Richards showed that ℓ^1 is not computably categorical [10]. It follows from a recent result of A.G. Melnikov that ℓ^2 is computably categorical [8]. At the 2014 Conference on Computability and Complexity in Analysis, A.G. Melnikov asked “For which computable reals $p \geq 1$ is ℓ^p computably categorical?” The following theorem answers this question.

Theorem 1.1. *Suppose p is a computable real so that $p \geq 1$. Then, ℓ^p is computably categorical if and only if $p = 2$.*

We prove Theorem 1.1 by proving the following stronger result.

Theorem 1.2. *Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a c.e. set. Then, there is a computable copy of ℓ^p , \mathcal{B} , so that C computes a linear isometry of ℓ^p onto \mathcal{B} . Furthermore, if an oracle X computes a linear isometry of ℓ^p onto \mathcal{B} , then X must also compute C .*

These results also hold for ℓ^p -spaces over the reals. In a forthcoming paper it will be shown that ℓ^p is Δ_2^0 -categorical.

The paper is organized as follows. Section 2 covers background and motivation. Section 3 presents the proof of Theorem 1.2. Concluding remarks are presented in Section 4.

2. BACKGROUND

2.1. Background from functional analysis. Fix p so that $1 \leq p < \infty$. A *generating set* for ℓ^p is a subset of ℓ^p with the property that ℓ^p is the closure of its linear span.

Let e_n be the vector in ℓ^p whose $(n+1)$ st component is 1 and whose other components are 0. Let $E = \{e_n : n \in \mathbb{N}\}$. We call E the *standard generating set* for ℓ^p .

Recall that an *isometry* of ℓ^p is a norm-preserving map of ℓ^p into ℓ^p . We will use the following classification of the surjective linear isometries of ℓ^p .

Theorem 2.1 (Banach/Lamperti). *Suppose p is a real number so that $p \geq 1$ and $p \neq 2$. Let T be a linear map of ℓ^p into ℓ^p . Then, the following are equivalent.*

- (1) *T is a surjective isometry.*
- (2) *There is a permutation of \mathbb{N} , ϕ , and a sequence of unimodular points, $\{\lambda_n\}_n$, so that $T(e_n) = \lambda_n e_{\phi(n)}$ for all n .*
- (3) *Each $T(e_n)$ is a unit vector and the supports of $T(e_n)$ and $T(e_m)$ are disjoint whenever $m \neq n$.*

In his seminal text on linear operators, S. Banach stated Theorem 2.1 for the case of ℓ^p spaces over the reals [2]. He also stated a classification of the linear isometries of $L^p[0, 1]$ in the real case. Banach's proofs of these results were sketchy and did not easily generalize to the complex case. In 1958, J. Lamperti rigorously proved a generalization of Banach's claims to real and complex L^p -spaces of σ -finite measure spaces [7]. Theorem 2.1 follows from J. Lamperti's work as it appears in Theorem 3.2.5 of [4]. Note that Theorem 2.1 fails when $p = 2$. For, ℓ^2 is a Hilbert space. So, if $\{f_0, f_1, \dots\}$ is any orthonormal basis for ℓ^2 , then there is a unique surjective linear isometry of ℓ^2 , T , so that $T(e_n) = f_n$ for all n .

2.2. Background from computable analysis. We assume the reader is familiar with the fundamental notions of computability theory as covered in [3].

Suppose $z_0 \in \mathbb{C}$. We say that z_0 is *computable* if there is an algorithm that given any $k \in \mathbb{N}$ as input computes a rational point q so that $|q - z_0| < 2^{-k}$. This is equivalent to saying that the real and imaginary parts of z_0 have computable decimal expansions.

Our approach to computability on ℓ^p is equivalent to the format in [10] wherein a more expansive treatment may be found.

Fix a computable real p so that $1 \leq p < \infty$. Let $F = \{f_0, f_1, \dots\}$ be a generating set for ℓ^p . We say that F is an *effective generating set* if there is an algorithm that given any rational points $\alpha_0, \dots, \alpha_M$ and a nonnegative integer k as input computes a rational number q so that

$$q - 2^{-k} < \left\| \sum_{j=0}^M \alpha_j f_j \right\| < q + 2^{-k}.$$

That is, the map

$$\alpha_0, \dots, \alpha_M \mapsto \left\| \sum_{j=0}^M \alpha_j f_j \right\|$$

is computable. Clearly the standard generating set is an effective generating set.

Suppose $F = \{f_0, f_1, \dots\}$ is an effective generating set for ℓ^p . We say that a vector $g \in \ell^p$ is *computable with respect to F* if there is an algorithm that given any nonnegative integer k as input computes rational points $\alpha_0, \dots, \alpha_M$ so that

$$\left\| g - \sum_{j=0}^M \alpha_j f_j \right\| < 2^{-k}.$$

Suppose $g_n \in \ell^p$ for all n . We say that $\{g_n\}_n$ is *computable with respect to F* if there is an algorithm that given any $k, n \in \mathbb{N}$ as input computes rational points $\alpha_0, \dots, \alpha_M$ so that

$$\left\| g_n - \sum_{j=0}^M \alpha_j f_j \right\| < 2^{-k}.$$

When $f \in \ell^p$ and $r > 0$, let $B(f; r)$ denote the open ball with center f and radius r . When $\alpha_0, \dots, \alpha_M$ are rational points and r is a positive rational number, we call $B\left(\sum_{j=0}^M \alpha_j f_j; r\right)$ a *rational ball*.

Suppose $F = \{f_0, f_1, \dots\}$ and $G = \{g_0, g_1, \dots\}$ are effective generating sets for ℓ^p . We say that a map $T : \ell^p \rightarrow \ell^p$ is *computable with respect to (F, G)* if there is an algorithm P that meets the following three criteria.

- **Approximation:** Given a rational ball $B(\sum_{j=0}^M \alpha_j f_j; r)$ as input, P either does not halt or produces a rational ball $B(\sum_{j=0}^N \beta_j g_j; r')$.
- **Correctness:** If $B(\sum_{j=0}^N \beta_j g_j; r')$ is the output of P on input $B(\sum_{j=0}^M \alpha_j f_j; r)$, then $T(f) \in B(\sum_{j=0}^N \beta_j g_j; r')$ whenever $f \in B(\sum_{j=0}^M \alpha_j f_j; r)$.
- **Convergence:** If U is a neighborhood of $T(f)$, then f belongs to a rational ball $B_1 = B(\sum_{j=0}^M \alpha_j f_j; r)$ so that P halts on B_1 and produces a rational ball that is included in U .

When we speak of an algorithm accepting a rational ball $B(\sum_{j=0}^M \alpha_j f_j; r)$ as input, we of course mean that it accepts some representation of the ball such as a code of the sequence $(r, M, \alpha_0, \dots, \alpha_M)$.

All of these definitions have natural relativizations. For example, if $F = \{f_0, f_1, \dots\}$ is an effective generating set, then we say that X computes a vector $g \in \ell^p$ with respect to F if there is a Turing reduction that given the oracle X and an input k computes rational points $\alpha_0, \dots, \alpha_M$ so that $\left\| g - \sum_{j=0}^M \alpha_j f_j \right\| < 2^{-k}$.

2.3. Background from computable categoricity. For the sake of motivation, we begin by considering the following simple example. Let ζ be an incomputable unimodular point in the plane. For each n , let $f_n = \zeta e_n$. Let $F = \{f_0, f_1, \dots\}$. Thus, F is an effective generating set. However, the vector ζe_0 is computable with respect to F even though it is not computable with respect to the standard generating set E . In fact, the only vector that is computable with respect to E and F is the zero vector. The moral of the story is that different effective generating sets may yield very different classes of computable vectors and sequences. However, there is a surjective linear isometry of ℓ^p that is computable with respect to (E, F) ; namely multiplication by ζ . Thus, E and F give the same computability theory on

ℓ^p even though they yield very different classes of computable vectors. This leads to the following definition.

Definition 2.2. Suppose p is a computable real so that $p \geq 1$. We say that ℓ^p is *computably categorical* if for every effective generating set F there is a surjective linear isometry of ℓ^p that is computable with respect to (E, F) .

The definitions just given for ℓ^p can easily be adapted to any separable Banach space. Suppose $G = \{g_0, g_1, \dots\}$ is an effective generating set for a Banach space \mathcal{B} . The pair (\mathcal{B}, G) is called a *computable Banach space*. Suppose that \mathcal{B} is linearly isometric to ℓ^p , and let T denote a linear isometric mapping of \mathcal{B} onto ℓ^p . Let $f_n = T(g_n)$, and let $F = \{f_0, f_1, \dots\}$. Then, F is an effective generating set for ℓ^p , and T is computable with respect to (G, F) . Thus, Theorem 1.2 can be rephrased as follows.

Theorem 2.3. *Suppose p is a computable real so that $p \geq 1$ and $p \neq 2$. Suppose C is a c.e. set. Then, there is an effective generating set for ℓ^p , F , so that with respect to (E, F) , C computes a surjective linear isometry of ℓ^p . Furthermore, any oracle that computes a surjective linear isometry of ℓ^p with respect to (E, F) must also compute C .*

A.G. Melnikov and K.M. Ng have investigated computable categoricity questions with regards to the space $C[0, 1]$ of continuous functions on the unit interval with the supremum norm [8], [9]. The study of computable categoricity for countable structures goes back at least as far as the work of Goncharov [5]. The text of Ash and Knight has a thorough discussion of the main results of this line of inquiry [1]. The survey by Harizanov covers other directions in the countable computable structures program [6].

3. PROOF OF THEOREMS 1.1 AND 1.2

We begin by noting the following easy consequence of the definitions and Theorem 2.1.

Proposition 3.1. *Suppose p is a computable real so that $p \geq 1$ and so that $p \neq 2$. Let F be an effective generating set for ℓ^p . Then, the following are equivalent.*

- (1) *There is a surjective linear isometry of ℓ^p that is computable with respect to (E, F) .*
- (2) *There is a permutation of \mathbb{N} , ϕ , and a sequence of unimodular points $\{\lambda_n\}_n$, so that $\{\lambda_n e_{\phi(n)}\}_n$ is computable with respect to F .*
- (3) *There is a sequence of unit vectors $\{g_n\}_n$ so that $\{g_n\}_n$ is computable with respect to F , $G = \{g_0, g_1, \dots\}$ is a generating set for ℓ^p , and so that the supports of g_n and g_m are disjoint whenever $n \neq m$.*

Proof. Parts (2) and (3) just restate each other. It follows from Theorem 2.1 that (1) implies (2).

Suppose (3) holds. Let T be the unique linear map of the span of E onto the span of G so that $T(e_n) = g_n$ for all n . Since the supports of g_0, g_1, \dots are pairwise disjoint, and since each g_n is a unit vector, T is isometric. It follows that there is a unique extension of T to a unique linear isometry of ℓ^p ; denote this extension by T as well. We claim that T is computable with respect to (E, F) . For, suppose a rational ball $B(\sum_{j=0}^M \alpha_j e_j; r)$ is given as input. Since $\{g_n\}_n$ is

computable with respect to F , it follows that we can compute a non-negative integer N and rational points β_0, \dots, β_N so that $\left\| \sum_{j=0}^M \alpha_j g_j - \sum_{j=0}^N \beta_j f_j \right\| < r$. We then output $B(\sum_{j=0}^N \beta_j g_j; 2r)$. It follows that the Approximation, Correctness, and Convergence criteria are satisfied and so T is computable with respect to (E, F) . \square

We now turn to the proof of Theorem 2.3 which, as we have noted, implies Theorem 1.2. Our construction of F is a modification of the construction used by Pour-El and Richards to show that ℓ^1 is not computably categorical [10]. Let C be an incomputable c.e. set. Without loss of generality, we assume $0 \notin C$. Let $\{c_n\}_{n \in \mathbb{N}}$ be an effective enumeration of C . Set

$$\gamma = \sum_{k \in C} 2^{-k}.$$

Thus, $0 < \gamma < 1$, and γ is an incomputable real. Set:

$$\begin{aligned} f_0 &= (1 - \gamma)^{1/p} e_0 + \sum_{n=0}^{\infty} 2^{-c_n/p} e_{n+1} \\ f_{n+1} &= e_{n+1} \\ F &= \{f_0, f_1, f_2, \dots\} \end{aligned}$$

Since $1 - \gamma > 0$, we can use the standard branch of $\sqrt[p]{\cdot}$.

We divide the rest of the proof into the following lemmas.

Lemma 3.2. *F is an effective generating set.*

Proof. Since

$$(1 - \gamma)^{1/p} e_0 = f_0 - \sum_{n=1}^{\infty} 2^{-c_{n-1}/p} f_n$$

the closed linear span of F includes E . Thus, F is a generating set for ℓ^p . Note that $\|f_0\| = 1$.

Suppose $\alpha_0, \dots, \alpha_M$ are rational points. When $1 \leq j \leq M$, set

$$\mathcal{E}_j = |\alpha_0 2^{-c_{j-1}/p} + \alpha_j|^p - |\alpha_0|^p 2^{-c_{j-1}}.$$

It follows that

$$\begin{aligned} \|\alpha_0 f_0 + \dots + \alpha_M f_M\|^p &= |\alpha_0|^p \|f_0\|^p + \mathcal{E}_1 + \dots + \mathcal{E}_M \\ &= |\alpha_0|^p + \mathcal{E}_1 + \dots + \mathcal{E}_M. \end{aligned}$$

Since $\mathcal{E}_1, \dots, \mathcal{E}_M$ can be computed from $\alpha_0, \dots, \alpha_M$, $\|\alpha_0 f_0 + \dots + \alpha_M f_M\|$ can be computed from $\alpha_0, \dots, \alpha_M$. Thus, F is an effective generating set. \square

Lemma 3.3. *Every oracle that with respect to F computes a scalar multiple of e_0 whose norm is 1 must also compute C .*

Proof. Suppose that with respect to F , X computes a vector of the form λe_0 where $|\lambda| = 1$. It suffices to show that X computes $(1 - \gamma)^{-1/p}$.

Fix a rational number q_0 so that $(1 - \gamma)^{-1/p} \leq q_0$. Let $k \in \mathbb{N}$ be given as input. Compute k' so that $2^{-k'} \leq q_0 2^{-k}$. Since X computes λe_0 with respect to F , we

can use oracle X to compute rational points $\alpha_0, \dots, \alpha_M$ so that

$$(3.1) \quad \left\| \lambda e_0 - \sum_{j=0}^M \alpha_j f_j \right\| < 2^{-k'}.$$

We claim that $|(1-\gamma)^{-1/p} - \alpha_0| < 2^{-k}$. For, it follows from (3.1) that $|\lambda - \alpha_0(1-\gamma)^{1/p}| < 2^{-k'}$. Thus, $|1 - |\alpha_0|(1-\gamma)^{1/p}| < 2^{-k'}$. Hence,

$$|(1-\gamma)^{-1/p} - \alpha_0| < 2^{-k'}(1-\gamma)^{-1/p} \leq 2^{-k'}q_0 \leq 2^{-k}.$$

Since X computes α_0 from k , X computes $(1-\gamma)^{-1/p}$. \square

Lemma 3.4. *If X computes a surjective linear isometry of ℓ^p with respect to (E, F) , then X must also compute C .*

Proof. By Lemma 3.3 and the relativization of Proposition 3.1. \square

Lemma 3.5. *With respect to F , C computes e_0 .*

Proof. Fix an integer M so that $(1-\gamma)^{-1/p} < M$.

Let $k \in \mathbb{N}$. Using oracle C , we can compute an integer N_1 so that $N_1 \geq 3$ and

$$\left\| \sum_{n=N_1}^{\infty} 2^{-c_{n-1}/p} e_n \right\| \leq \frac{2^{-(kp+1)/p}}{2^{-(kp+1)/p} + M}.$$

We can use oracle C to compute a rational number q_1 so that $|q_1 - (1-\gamma)^{-1/p}| \leq 2^{-(kp+1)/p}$. Set

$$g = q_1 \left[f_0 - \sum_{n=1}^{N_1-1} 2^{-c_{n-1}/p} f_n \right].$$

It suffices to show that $\|e_0 - g\| < 2^{-k}$. Note that since $1-\gamma < 1$, $|q_1(1-\gamma)^{1/p} - 1| \leq 2^{-(kp+1)/p}$. Note also that $|q_1| < M + 2^{-(kp+1)/p}$. Thus,

$$\begin{aligned} \|e_0 - g\|^p &= \left\| e_0 - q_1(1-\gamma)^{1/p} e_0 - q_1 \sum_{n=N_1}^{\infty} 2^{-c_{n-1}/p} e_n \right\|^p \\ &\leq |q_1(1-\gamma)^{1/p} - 1|^p + |q_1|^p \left\| \sum_{n=N_1}^{\infty} 2^{-c_{n-1}/p} e_n \right\|^p \\ &< 2^{-(kp+1)} + 2^{-(kp+1)} = 2^{-kp} \end{aligned}$$

Thus, $\|e_0 - g\| < 2^{-k}$. This completes the proof of the lemma. \square

Lemma 3.6. *With respect to (E, F) , C computes a surjective linear isometry of ℓ^p .*

Proof. By Lemma 3.5 and the relativization of Proposition 3.1. \square

4. CONCLUDING REMARKS

We note that all of the steps in the above proofs work just as well over the real field.

Lamperti's result on the isometries of L^p spaces hold when $0 < p < 1$. For these values of p , ℓ^p is a metric space under the metric

$$d(\{a_n\}_n, \{b_n\}_n) = \sum_{n=0}^{\infty} |a_n - b_n|^p.$$

The steps in the above proofs can be adapted to these values of p as well.

In a forthcoming paper it will be shown that ℓ^p is Δ_2^0 -categorical.

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